Indian Statistical Institute
Semestral Examination
Differential Geometry II - BMath III

Max Marks: 60 Time: 180 minutes.

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

(1) (a) Construct a smooth map $f : S^2 \longrightarrow S^1$ having exactly two critical points.
(b) Construct an embedding $f : S^1 \longrightarrow \mathrm{GL}_2(\mathbb{R})$.
(c) Show that S^1 is diffeomorphic to $\mathbb{RP}^1$.
(d) Give an example of a nowhere zero n-form on S^n.
(e) Give examples to show that the sum and Lie-bracket of complete vector fields need not be complete.
(f) Describe the geodesics of $(\mathbb{R}^n, \mathrm{Can})$. $[6 \times 4 = 24]$

(2) (a) Define the notion of orientability of manifolds. Show that $\mathbb{RP}^n$ is orientable for odd n.
(b) Compute $H^i_{dR}(S^1)$. $[8 + 10 = 18]$

(3) (a) Show that the map $f : (\mathbb{H}, g) \longrightarrow (\mathbb{D}, g')$ defined by

$$f(z) = \frac{z-i}{z+i}$$

is an isometry where

$$g_{(x,y)} = \frac{dx^2 + dy^2}{y^2}; \; g'_{(x,y)} = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}.$$

Here, as usual, $\mathbb{H}$ denotes the upper half-plane and $\mathbb{D}$ the open unit disc in the plane.
(b) Define the term : Levi-Civita connection. Describe the Levi-Civita connection on the upper half-plane with the Poincare metric.
(c) Define the notion of parallel transport. Let ∇ be the connection on $\mathbb{R}^2$ determined by the Christoffels

$$\Gamma^k_{ij} = \begin{cases} x & \text{if } (i,j,k) = (2,2,1) \\ 0 & \text{otherwise} \end{cases}$$

Let $\sigma, \theta : [0,1] \longrightarrow \mathbb{R}^2$ be the paths defined by $\sigma(t) = (t,t)$ and $\theta(t) = (t, t^2)$. Compute the parallel transport maps P_{01} corresponding to σ and θ from $T_{\sigma(0)}(\mathbb{R}^2) \longrightarrow T_{\theta(1)}(\mathbb{R}^2)$.
$[6 \times 3 = 18]$